Chapter 3.

Vectors – basic ideas.

Situation One

A mother duck shelters by the bank of a pond with her young family of ducklings. One brave duckling ventures out for a swim. The duckling swims 5 metres on a bearing 050° followed by 7 metres on a bearing 170°.

- How far is the duckling from its mother at the end of this two stage excursion?
- What is the bearing of the duckling from the mother?
- What is the bearing of the mother from the duckling?

Situation Two

An orienteering competition is to involve ten stages. The organisers print the instuctions for travelling from each check point to the next. These instructions commence as follows:

From the start travel 400 metres on a bearing 040° to 1st checkpoint.
From 1st checkpoint travel 300 metres on a bearing 100° to 2nd checkpoint.
From 2nd checkpoint travel 450 metres on a bearing 180° to 3rd checkpoint.

The organisers are concerned that if heavy rain falls the night before the event then checkpoint 2 will be unapproachable. In case of this they need alternative instructions for a competition involving nine stages. This alternative competition would not involve the original checkpoint 2 but re-numbers checkpoints 3 to 10 as 2 to 9. Copy and complete the following instructions for the first two stages of this alternative race.

From the start travel _____ metres on a bearing _______ to 1st checkpoint.
From 1st checkpoint travel _____ metres on a bearing _______ to 2nd checkpoint.

Vector quantities.

Each of the situations on the previous page involved the addition of two stages of a journey. Hopefully you did not simply add the two distances together to give the final distance from the starting point. Such an approach would not give the correct answer because, whilst 5 m + 7 m = 12 m

5 m in direction 050° + 7 m in direction 170° ≠ 12 m in a direction 220°.

Our normal rules for the addition of two quantities will not necessarily hold when the two quantities to be added involve direction. For example, if a person walks 5 km due North followed by 3 km due South the total distance walked will be 8 km but the person will not be 8 km from the starting point.

1st leg of journey	2nd leg of journey	Dist travelled	Final location.
5 km N	3 km S	8 km	2 km N of starting point.

In this case the directions of each leg of the journey are such that the final location could be determined mentally. If this is not the case we must either

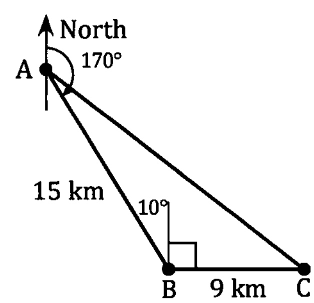
- sketch the situation and use trigonometry to determine the final location relative to the initial position,

or
- make an accurate scale drawing to determine the final location relative to the initial position.

Example 1

A boat sails 15 km on a bearing 170° followed by 9 km due East. Find the distance and bearing of the boat's final position from its initial position.

Method One: By calculation (i.e. sketch and use trigonometry).



If A is the initial position and C is the final position then the distance from A to C is given by AC.

Now $AC^2 = 15^2 + 9^2 - 2 \times 15 \times 9 \times \cos 100°$ (cos rule)

$\therefore \quad AC \approx 18{\cdot}79$

The bearing of C from A is $(170° - \angle BAC)$

By the sine rule $\dfrac{9}{\sin \angle BAC} = \dfrac{AC}{\sin 100°}$

giving $\angle BAC \approx 28{\cdot}2°$ Obtuse solution to equation not applicable as triangle already has an obtuse angle.

The final position of the boat is approximately 19 km from the initial position and on a bearing 142°.

Important note: Had there not already been an obtuse angle in ΔBAC, finding ∠BAC, rather than ∠ACB, is a wise strategy because ∠BAC is not opposite the longer side and so cannot be obtuse.

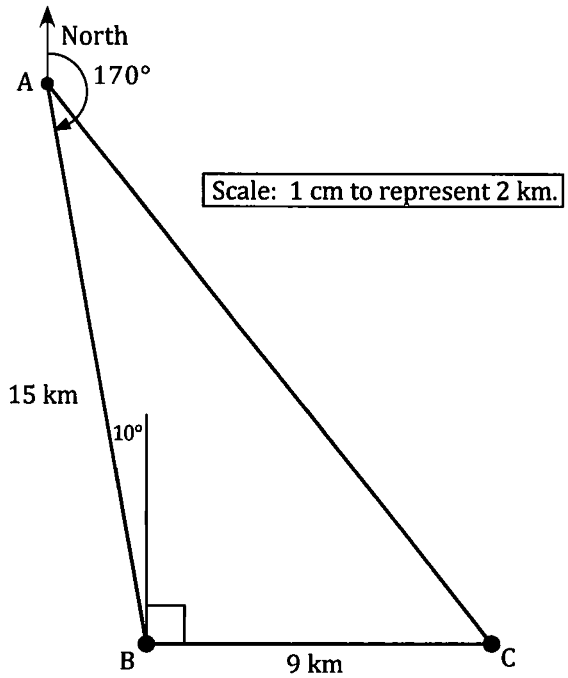
<u>Method Two</u>: By scale drawing.

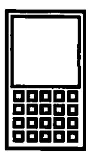
If A is the initial position and C is the final position then the distance from A to C is given by AC.

From the scale drawing shown on the right:

$$AC \approx 9{\cdot}4 \text{ cm}$$
and $$\angle BAC \approx 28°.$$

The final position of the boat is approximately 19 km from the initial position and on a bearing 142°.





Rather than creating the scale drawing with pencil, ruler and protractor it is possible to create it using the drawing facility of some calculators. Explore the capability of your calculator in this regard.

As we have seen above, adding quantities which have magnitude and direction needs special care. Such quantities are called **vector** quantities. Some common examples of vector quantities are:

Displacement,	e.g. 5 km South.
Velocity,	e.g. 5 m/s North.
Force,	e.g. 6 Newtons upwards.
Acceleration,	e.g. 5 m/s^2 East.

Quantities which have magnitude only are not vectors. Such quantities are called **scalars**. Some common examples of scalar quantities are:

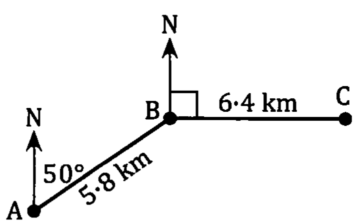
Distance,	e.g. 5 km.
Speed,	e.g. 5 m/s.
The magnitude of a force,	e.g. 6 Newtons.
The magnitude of the acceleration,	e.g. 5 m/s^2.
Energy,	e.g. 50 Joules.

Exercise 3A

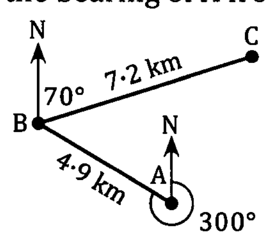
Questions 1, 2 and 3 each show a sketch of a journey from A to C via B.

Use trigonometry to calculate (a) the distance and bearing of C from A,
(b) the bearing of A from C.

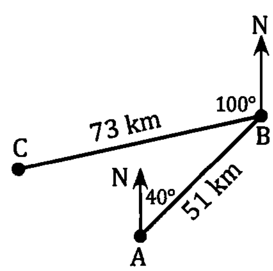
1.



2.



3.

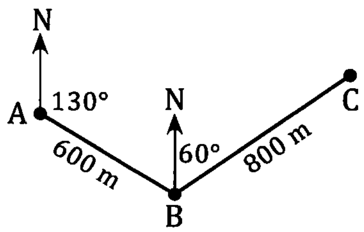


Questions 4, 5 and 6 each show a sketch of a journey from A to C via B.
Using an accurate scale drawing for each case determine

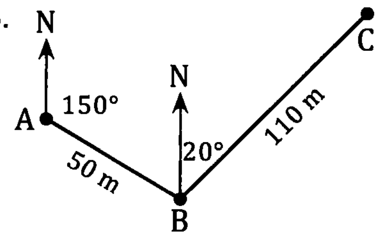
(a) the distance and bearing of C from A,

(b) the bearing of A from C.

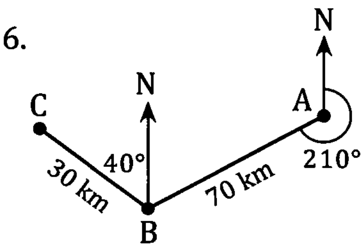
4.



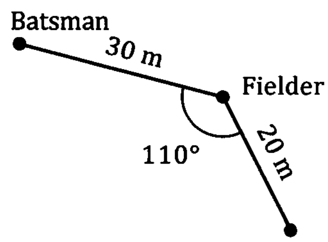
5.



6.



7. In a cricket match a batsman hits the ball and a fielder, 30 m away, fails to stop the ball but deflects it (see diagram) and slows it down. The ball comes to rest 20 m from where the fielder deflected it.
Find the distance from where the ball was hit to where it came to rest.

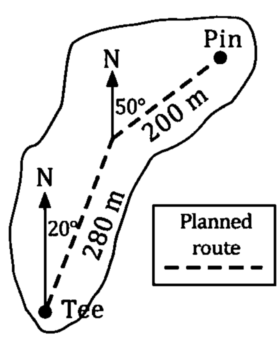


8. A yacht sails 5·2 km on a bearing 190° followed by 6·4 km on a bearing 110°. Calculate the distance and bearing of the yacht's final position from its initial position.

9. A hiker travels 2·6 km due North followed by 4·3 km on a bearing 132°. Calculate the distance and bearing of the hiker's final position from his initial position.

10. In an orienteering race the first checkpoint is 500 m from the start and on a bearing of 030°. The second checkpoint is 400 m from the first checkpoint and is 600 m from the start. Find, to the nearest degree, the two possible bearings of checkpoint 2 from the start.

11. A particular hole on a golf course is as shown on the right. A positive thinking golfer plans to complete the hole in two shots as shown by the "planned route". His first shot is to be 280 m at 020° and the second 200 m at 050°.
Unfortunately he miss-hits his first shot and it goes 250 m due North.
Calculate the distance (nearest metre) and direction (nearest degree) that his second shot needs to be if he is still to complete the hole in two shots.



As was explained on the previous page, quantities which have magnitude and direction are called **vector** quantities and special care needs to be taken when such quantities are added. The situations at the beginning of this chapter, example one and all of the questions in Exercise 3A involved the vector quantity **displacement**. The situation on the next page involves the vector quantity **force**.

Situation Three

Each of the following diagrams show one or two tugboats about to tow a floating platform into position. The force that each tug is exerting, in Newtons, N, is as indicated. Find the direction in which the platform will begin to move in each case, giving your answer as a bearing, and state the total force acting in that direction.

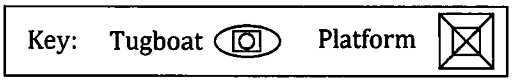


DIAGRAM 1

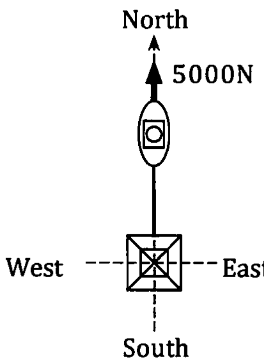


DIAGRAM 2

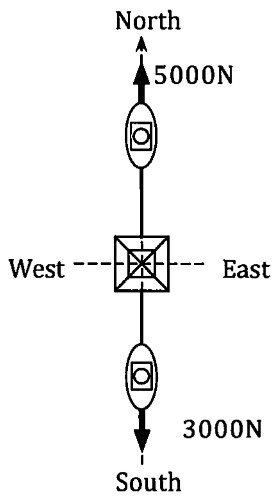


DIAGRAM 3

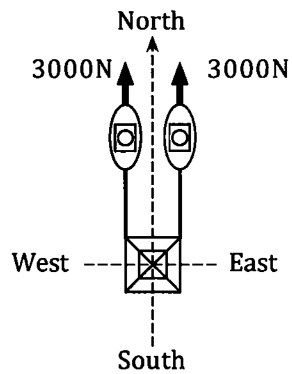


DIAGRAM 4

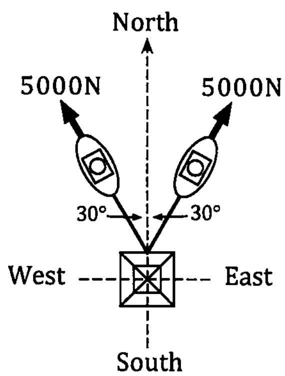


DIAGRAM 5

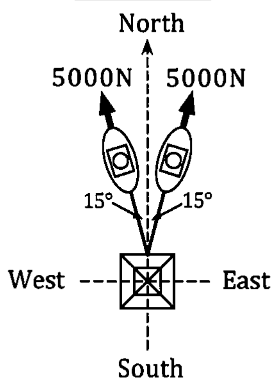


DIAGRAM 6

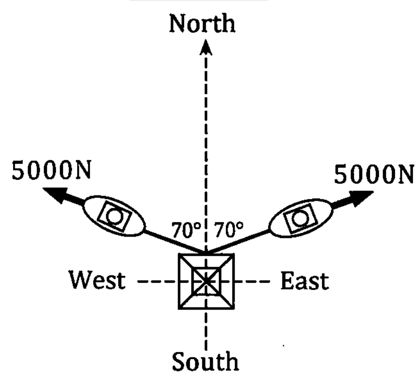


DIAGRAM 7

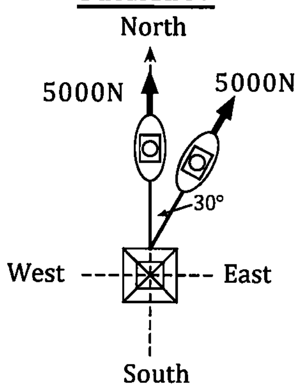


DIAGRAM 8

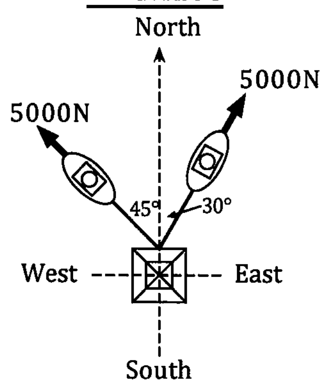
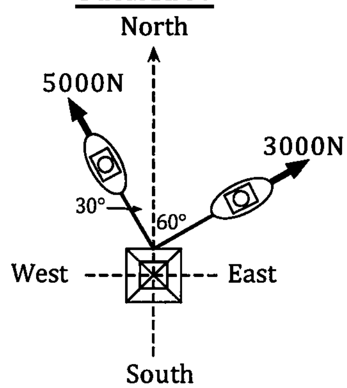


DIAGRAM 9



Adding Vectors.

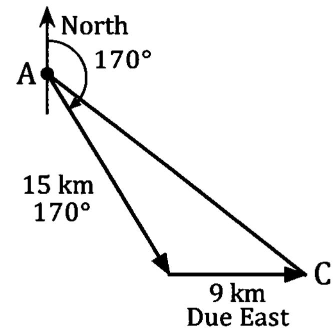
Situation three involved the addition of vector quantities, in each case two forces. Diagrams 1, 2 and 3 should not have caused you any problems but how did you get on with diagrams 4 to 9? Did you develop any methods for adding two vector quantities, (in this case forces)?

As has been previously mentioned care needs to be taken when adding vectors because we cannot simply sum the magnitudes and sum the directions:

5 N in direction 050° + 7 N in direction 170° ≠ 12 N in a direction 220°.

In example 1, encountered earlier, we added displacement vectors by sketching and using trigonometry or by scale drawing.

To determine the combined or **resultant** effect of 15 km in a direction 170° and 9 km due East we made a sketch like that shown on the right and then used trigonometry to determine the magnitude and direction of AC.



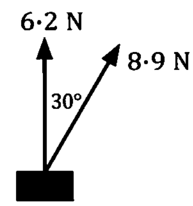
Notice that the two vectors to be added are drawn "nose to tail", i.e. the "tail" of the second vector takes over from the "nose" of the first.

This was clearly the logical thing to do with displacement vectors because the second leg of the journey did indeed start from where the first leg finished. However we can use this same "nose to tail" method to determine the resultant of vector quantities other than displacement.

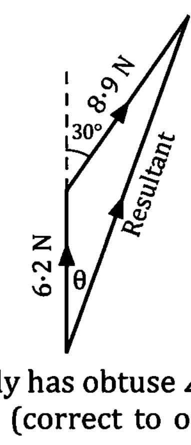
Note also in the above diagram that the resultant vector is given by the magnitude and direction of the line AC, the side that completes the triangle from the tail of the first vector to the tip of the second vector.

Example 2

Forces of 6·2 Newtons vertically upwards and 8·9 Newtons acting at 30° to the vertical act on a body, see diagram. Determine the magnitude and direction of the single force that could replace these two forces, (i.e. determine the **resultant** of the two forces).



First sketch the vector triangle remembering the "nose to tail" idea for vector addition.



By the cosine rule:

$$(\text{Magnitude of resultant})^2 = 6{\cdot}2^2 + 8{\cdot}9^2 - 2(6{\cdot}2)(8{\cdot}9)\cos 150°$$

$$\therefore \quad \text{Magnitude of resultant} \approx 14{\cdot}602 \text{ Newtons}$$

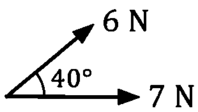
By the sine rule: $$\frac{\text{Magnitude of resultant}}{\sin 150°} = \frac{8{\cdot}9}{\sin\theta}$$

$$\therefore \quad \theta \approx 17{\cdot}7°$$

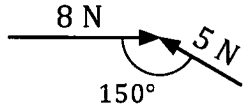
(Obtuse solution not applicable, θ not opposite longest side, and Δ already has obtuse ∠.)

The resultant of the two forces is a force of magnitude 14·6 Newtons (correct to one decimal place) acting at 18° to the vertical (nearest degree).

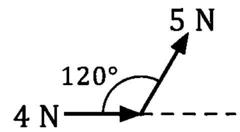
Some questions refer to the angle between two forces. This refers to the angle between the forces when they are either both directed away from a point or both directed towards it.



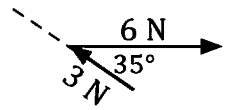
Angle between forces is 40°.



Angle between forces is 150°.



Angle between forces is 60°.

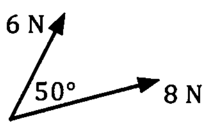


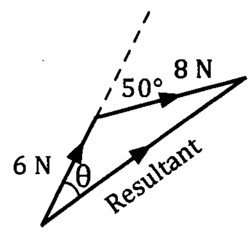
Angle between forces is 145°.

Example 3

Two forces have magnitudes of 6 N and 8 N and the angle between them is 50°. Find the magnitude of the resultant and the angle it makes with the smaller of the two forces.

First sketch the situation and then draw the vector triangle:





By the cosine rule: Magnitude of resultant $= \sqrt{6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 130°}$

$\approx 12{\cdot}7$

By the sine rule: $\dfrac{\text{Magnitude of resultant}}{\sin 130°} = \dfrac{8}{\sin \theta}$

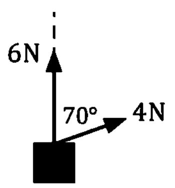
$\theta \approx 28{\cdot}8°$

The resultant of the two forces has magnitude 12·7 N and makes an angle of approximately 29° with the smaller of the two forces.

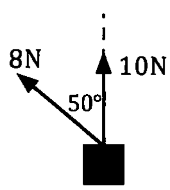
Exercise 3B

Questions 1 to 4 each show two forces acting on a body. In each case determine the magnitude, correct to one decimal place, and direction, to the nearest degree, of the resultant of the two forces. (Give the direction in terms of the angle made with the vertical - shown as a broken line).

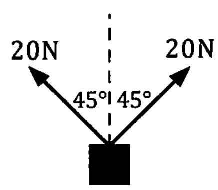
1.



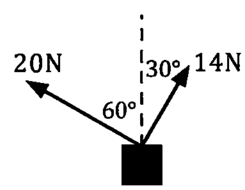
2.



3.

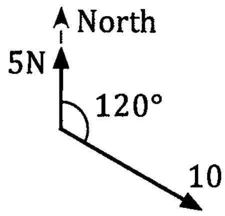


4.

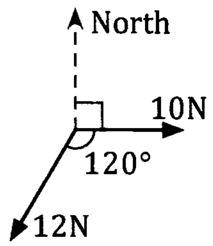


Find the magnitude and direction of the resultant of each of the following pairs of forces. (Give magnitude in exact form for questions 5 and 6 and correct to one decimal place for 7 and 8 and directions as three figure bearings to the nearest degree.)

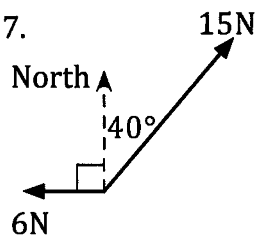
5.



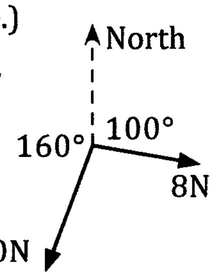
6.



7.

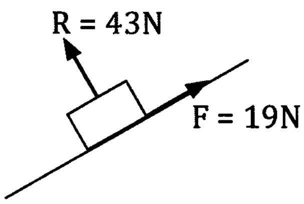


8.

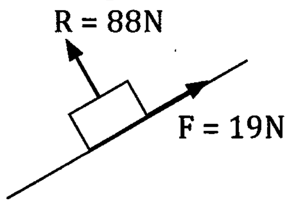


When a body slides down a rough inclined plane it experiences a normal reaction, R, perpendicular to the surface and a frictional force F, parallel to the surface. The resultant of R and F is called the total reaction. Determine the magnitude (nearest Newton) and direction (nearest degree) of the total reaction in each of questions 9, 10 and 11 giving the direction as the acute angle the total reaction makes with the slope.

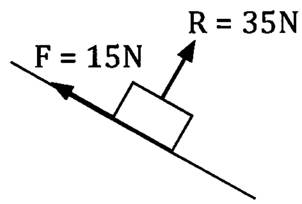
9.



10.



11.



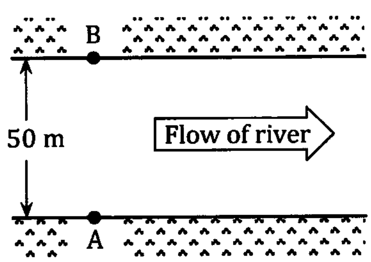
12. Two forces have magnitudes of 8 N and 12 N and the angle between them is 130°. Find the magnitude of the resultant and the angle it makes with the larger of the two forces.

13. Two forces have magnitudes of 10 N and 15 N and the angle between them is 45°. Find the magnitude of the resultant and the angle it makes with the smaller of the two forces.

We have so far used this "nose to tail" approach for adding two displacement vectors and for adding two forces. The following example, and Exercise 3C that follows it, involve this same process used for velocity vectors.

Example 4

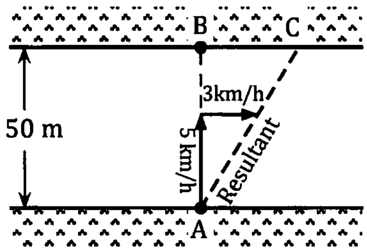
A canoeist wishes to paddle her canoe across a river from point A on one bank to the opposite bank. The canoeist can maintain a constant 5 km/h in still water. However the river is flowing at 3 km/h (see diagram).



(a) If the canoeist wishes to journey across the river as quickly as possible how long will the journey take and how far down river will the canoeist travel?

(b) If instead she wishes to reach point B on the opposite bank, directly opposite A, in which direction should she paddle and how long will the journey take?

(a) If she wishes to journey across the river as quickly as possible she must put all her efforts into getting across and let the current take her down river.
She travels at 5 km/h across the river and the current takes her down river at 3 km/h. Her resultant velocity is as shown on the right.



Now speed = $\dfrac{\text{distance}}{\text{time}}$.

Thus to travel the 50 m across the river when her speed across is 5 km/h will take t_1 seconds where

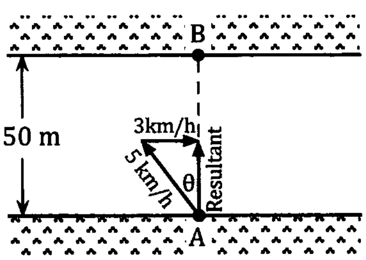
$$\frac{5 \times 1000}{60 \times 60} = \frac{50}{t_1} \qquad \therefore \quad t_1 = 36.$$

The distance travelled downstream will be given by BC, see diagram.

By similar triangles $\qquad \dfrac{3}{5} = \dfrac{BC}{50} \qquad \therefore \quad BC = 30.$

If she wishes to journey across the river as quickly as possible it will take her 36 seconds and she will travel 30 metres down river.

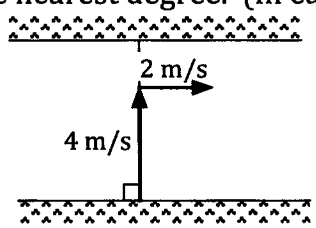
(b) To reach point B on the far bank she needs to set a course such that the combined effect of her paddling and the flow of the river takes her directly across the river (see diagram).
By trigonometry: $\theta \approx 37°$
By Pythagoras: Resultant speed = 4 km/h.
To travel 50 metres at 4 km/h will take 45 secs.

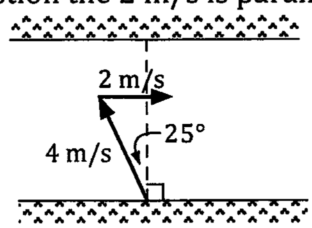


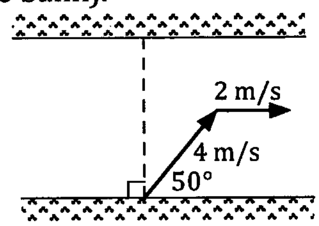
To reach point B on the opposite bank, directly opposite A, she should paddle upstream at 53° to the bank. The combined effect of this paddling and the current will cause her to travel to B in 45 seconds.

Exercise 3C

Find the resultant of each of the following pairs of velocities stating the magnitude of the resultant, correct to one decimal place, and the acute angle it makes with the bank, to the nearest degree. (In each question the 2 m/s is parallel to the bank).

1. 

2. 

3. 

4. A boat starts out at 20 km/h in direction 030°. However the boat is blown off this direction by a 12 km/h wind blowing from 080°. What direction does the boat now travel in and how far does it travel in one hour?

5. A bird flying south at 50 km/h encounters a 24 km/h wind from 330°. In what direction and with what speed will the bird now be travelling?
If the bird wishes to continue flying south, in what direction should it attempt to fly so that the wind causes it to travel south?

6. A hot-air balloon is gaining height at 3 m/s and the wind is blowing horizontally at a steady 1 m/s.
Find (a) the height of the balloon one minute after take off,
(b) the speed of the balloon,
(c) the angle the balloon's travel makes with the horizontal.

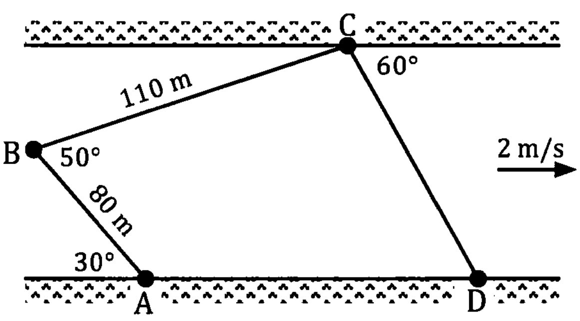
7. A boat has a speed of 10 km/h in still water. It is to be driven directly across a river of width 80 m. At what angle to the bank must the boat be directed, to the nearest degree, and how long will the journey take, to the nearest second, if the current flows at (a) 3 km/h, (b) 4 km/h, (c) 6 km/h.

8. In still air an aircraft can maintain a speed of 400 km/h. In what direction should the aircraft be pointing if it is to travel due north and a wind of 28 km/h is blowing from the west.

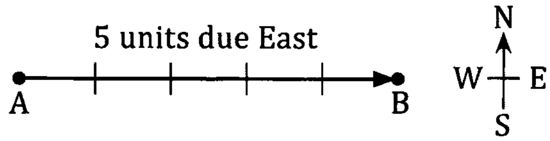
9. In still air an aircraft can maintain a speed of 300 km/h. In what direction should the aircraft be pointing if it is to travel due north and a 28 km/h wind is blowing from 070°.

10. In what direction must a plane head if it can maintain 350 km/h in still air, wishes to fly 500 km in a direction 040° and the wind is blowing at 56 km/h from 100°? How long would the journey take (to the nearest minute)?
How long would the return journey take, again to the nearest minute, if the wind is still 56 km/h from 100°?

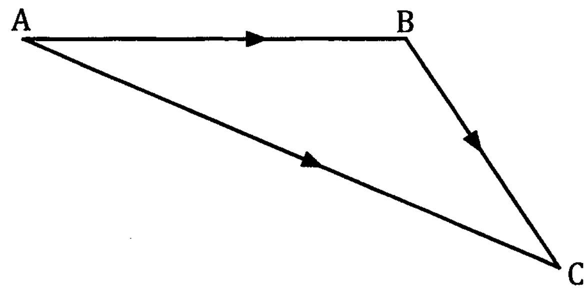
11. The diagram shows a canoeing course in a river. The canoeist has to go from the start at A to B to C to the finish at D. The river is flowing at 2 m/s and a particular canoeist can paddle at a rate that would produce a speed of 6 m/s in still water. Find, to the nearest second, the least time this canoeist takes to complete the course.



Mathematical representation of a vector quantity.
The vectors considered so far in this chapter have been displacement, force and velocity. When considering the manipulation of vector quantities we need not have a particular vector context, such as force or velocity, in mind. We could simply consider a vector of given magnitude, say 5 units, and with given direction, say due East. We represent such a vector diagrammatically by a line segment in the given direction and whose length represents the magnitude of the vector.

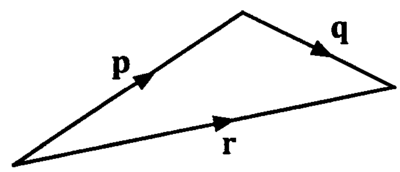


The above vector is from A to B and we write it as $\underline{AB}$ or $\overrightarrow{AB}$. The order of the letters indicates the direction, i.e. **from** A **to** B. The arrow above the letters emphasises this direction and this arrow, or the line underneath, distinguishes the vector $\underline{AB}$ from the scalar AB, the distance from A to B.



From our "nose to tail" method of addition it follows from the diagram on the right that:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



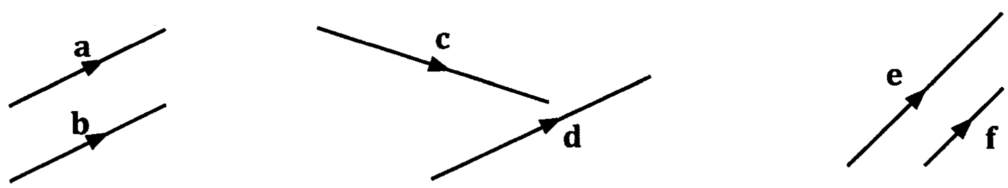
Vectors are also frequently written using lower case letters and in such cases bold type is used. Thus from the diagram on the right it follows that:

$$\mathbf{p} + \mathbf{q} = \mathbf{r}.$$

This use of bold single letters for vectors is fine in a text book but it is not easy to show bold lettering when writing by hand. Thus when writing by hand it is usual to use underlining to represent the vector and to write **a** as $\underline{a}$, **b** as $\underline{b}$, etc.

For the magnitude of vector **a** we write $|\mathbf{a}|$ or $|\underline{a}|$.

Equal vectors.
Two vectors are equal if they have the same magnitude and the same direction.

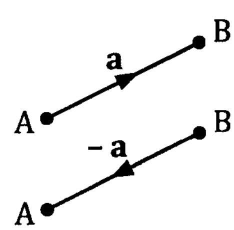


$\mathbf{a} = \mathbf{b}$ Same magnitude, same direction.

$\mathbf{c} \neq \mathbf{d}$ Same magnitude, different directions.

$\mathbf{e} \neq \mathbf{f}$ Same direction, different magnitudes.

The negative of a vector.
The vectors, **a** and –**a**, will have the same magnitudes but will be in opposite directions.



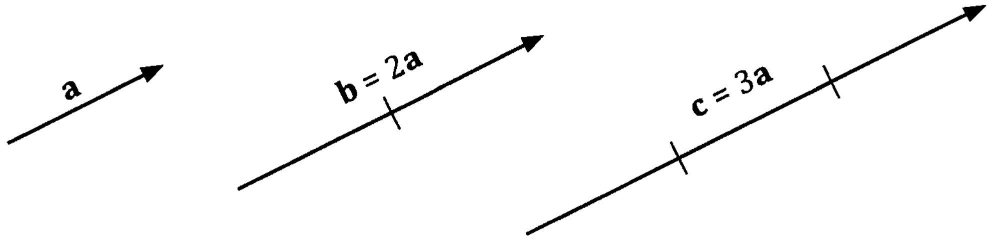
If $\mathbf{a} = \overrightarrow{AB}$ then $-\mathbf{a} = -\overrightarrow{AB}$

$$= \overrightarrow{BA}$$

ISBN 9780170350525.

Multiplication of a vector by a scalar.

If **b** = 2**a** then **b** is in the same direction as **a** but twice the magnitude.
If **c** = 3**a** then **c** is in the same direction as **a** but three times the magnitude.

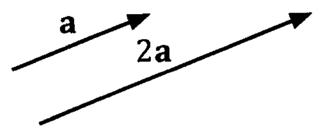


If for some positive scalar k, **b** = k**a**, then **b** is in the same direction as **a** and k times the magnitude.
If k is negative then **b** will be in the opposite direction to **a** and k times the magnitude.

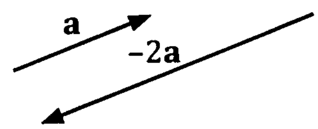
Parallel vectors.

Two vectors are parallel if one is a scalar multiple of the other.

If the scalar multiple is positive the vectors are said to be like parallel vectors, i.e. in the same direction.
e.g. **a** and 2**a** are like parallel vectors.



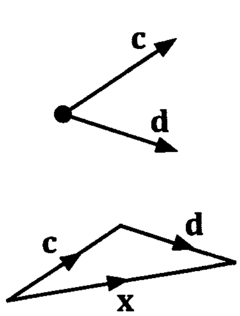
If the scalar multiple is negative the vectors are said to be unlike parallel vectors, i.e. in opposite directions.
e.g. **a** and −2**a** are unlike parallel vectors.



Addition of vectors.

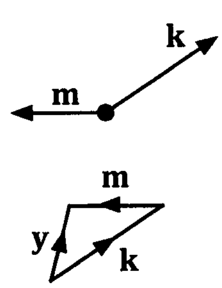
To add two vectors means to find the single, or resultant, vector that could replace the two. We add the vectors using a vector triangle in which the two vectors to be added follow "nose to tail" and form two of the sides of the triangle. The resultant is then the third side of the triangle with its direction "around the triangle" being in the opposite sense to the other two.

To add **c** and **d**



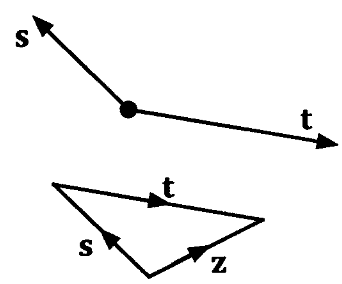
x = **c** + **d**

To add **k** and **m**



y = **k** + **m**

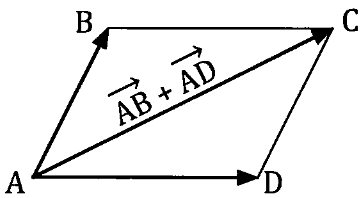
To add **s** and **t**



z = **s** + **t**

This vector addition is sometimes referred to as the *parallelogram law*:

> If the two vectors to be added are represented in magnitude and direction by $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in the parallelogram ABCD then the resultant of the two vectors will be represented in magnitude and direction by the diagonal $\overrightarrow{AC}$.

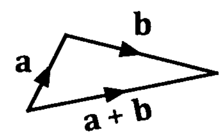


Considering the triangle ADC and remembering that AB and DC are opposite sides of a parallelogram, i.e. $\overrightarrow{AB} = \overrightarrow{DC}$, it can be seen that the parallelogram method of vector addition and the triangle method are equivalent.

With $|\mathbf{a}|$ as the magnitude of **a** it also follows that

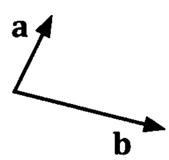
$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

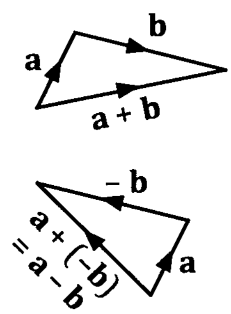
the triangle inequality.



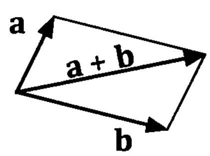
Subtraction of one vector from another.

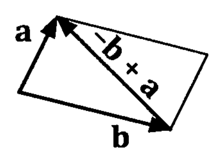
To give meaning to vector subtraction we consider **a** – **b** as **a** + (–**b**) and then use our technique for adding vectors.





With the parallelogram approach we see that whilst **a** + **b** is one diagonal, **a** – **b** is the other.





The zero vector.

If we add a vector to the negative of itself we obtain the zero vector: $\mathbf{p} + (-\mathbf{p}) = \mathbf{0}$.
The zero vector has zero magnitude and an undefined direction. We can write the zero vector as **0** or $\underline{0}$ or simply as 0 because it is usually clear from the context whether 0 is a number or a vector.

$h\mathbf{a} = k\mathbf{b}$.

The vector statement $h\mathbf{a} = k\mathbf{b}$ means that some scalar multiple of **a** has the same magnitude and direction as some scalar multiple of **b**.

If this is the case then either
- **a** and **b** are parallel vectors, (because one is a scalar multiple of the other),

or
- $h = k = 0$.

Thus if **a** and **b** are <u>not</u> parallel and we have a vector expression of the form

$$p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \qquad ①$$

i.e. $$(p - r)\mathbf{a} = (s - q)\mathbf{b},$$

it follows that $p - r = 0$ and $s - q = 0$.

i.e. $p = r$ and $s = q$.

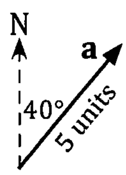
Thus in equation ①, with **a** and **b** not parallel, we can equate the coefficients of **a** to give $p = r$, and we can equate the coefficients of **b** to give $q = s$.

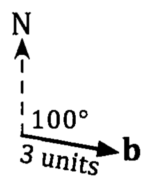
Now that we are able to represent a vector mathematically using line segments and letters we are able to consider abstract vector questions in which the context e.g. velocity, force etc. is not known. Examples 5 and 6 demonstrate this manipulation of abstract vectors.

Example 5

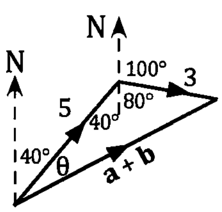
If **a** is a vector of magnitude 5 units in direction 040° and **b** is a vector of magnitude 3 units in direction 100° find the magnitude and direction of (a) **a** + **b**, (b) **a** − **b**.

First sketch **a** and **b**:





(a)



By the cosine rule $\quad |\mathbf{a} + \mathbf{b}|^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos 120°$

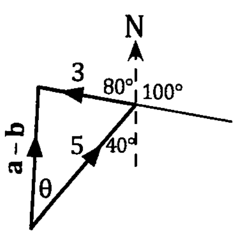
Thus $\quad |\mathbf{a} + \mathbf{b}| = 7$

By the sine rule $\quad \dfrac{|\mathbf{a} + \mathbf{b}|}{\sin 120°} = \dfrac{3}{\sin \theta}$

Thus $\quad \theta \approx 21{\cdot}8°$

a + **b** has a magnitude of 7 units and direction 062° (to the nearest degree).

(b)



By the cosine rule $\quad |\mathbf{a} - \mathbf{b}|^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos 60°$

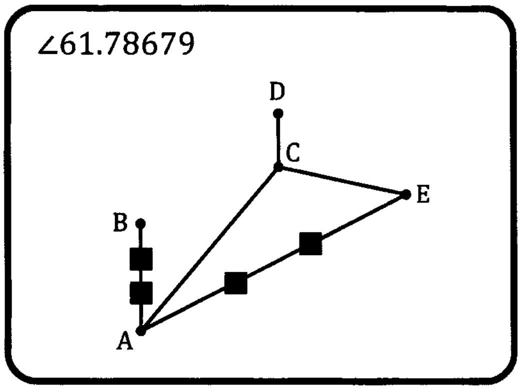
Thus $\quad |\mathbf{a} - \mathbf{b}| \approx 4{\cdot}36$

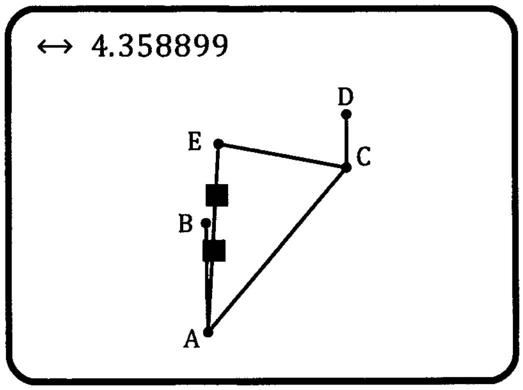
By the sine rule $\quad \dfrac{|\mathbf{a} - \mathbf{b}|}{\sin 60°} = \dfrac{3}{\sin \theta}$

Thus $\quad \theta \approx 36{\cdot}6°$

a − **b** has magnitude 4·4 units (correct to 1 d.p.) and direction 003° (to nearest degree).

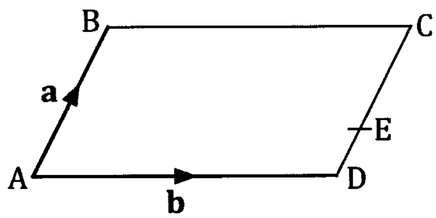
Alternatively scale versions of the triangles used to determine **a** + **b** and **a** – **b** could be drawn by hand or using the ability of some calculators to draw geometrical figures. Again you are encouraged to explore the capability of your calculator in this regard.





Example 6

In parallelogram ABCD, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.
E is a point on DC such that DE : EC = 1 : 2.
Express each of the following vectors in terms of **a** and/or **b**.



(a) $\overrightarrow{DC}$, (b) $\overrightarrow{CB}$, (c) $\overrightarrow{DE}$, (d) $\overrightarrow{BE}$.

(a) $\overrightarrow{DC}$ is the same magnitude and direction as $\overrightarrow{AB}$. Thus $\overrightarrow{DC} = \mathbf{a}$.

(b) $\overrightarrow{CB}$ is the same magnitude but opposite direction to $\overrightarrow{AD}$. Thus $\overrightarrow{CB} = -\mathbf{b}$.

(c) DE : EC = 1 : 2. Thus $\overrightarrow{DE} = \frac{1}{3}\overrightarrow{DC}$

$$\therefore \quad \overrightarrow{DE} = \frac{1}{3}\mathbf{a}$$

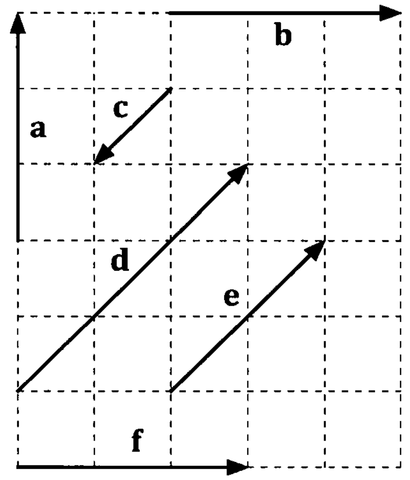
(d)
$$\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{CE} = \mathbf{b} + \left(-\frac{2}{3}\mathbf{a}\right) = \mathbf{b} - \frac{2}{3}\mathbf{a}$$

or

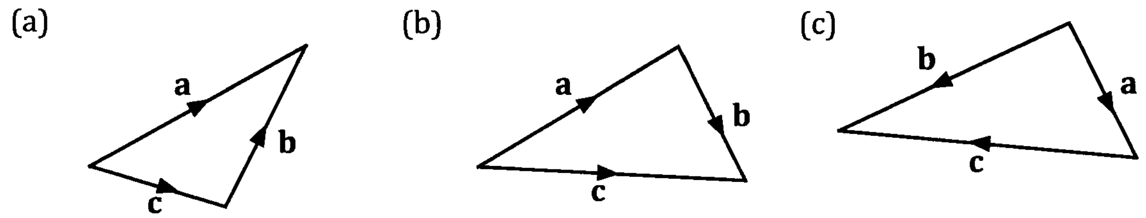
$$\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DE} = -\mathbf{a} + \mathbf{b} + \frac{1}{3}\mathbf{a} = -\frac{2}{3}\mathbf{a} + \mathbf{b}$$

Exercise 3D

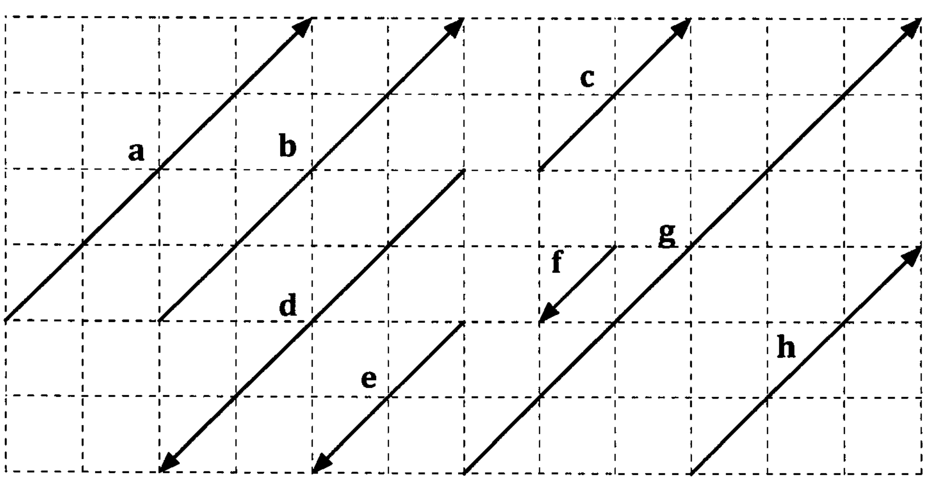
1. For the vectors **a** to **f** shown on the right state
   (a) two like parallel vectors that are unequal,
   (b) two unlike parallel vectors that are unequal,
   (c) two vectors that are the same magnitude but not equal,
   (d) two equal vectors.



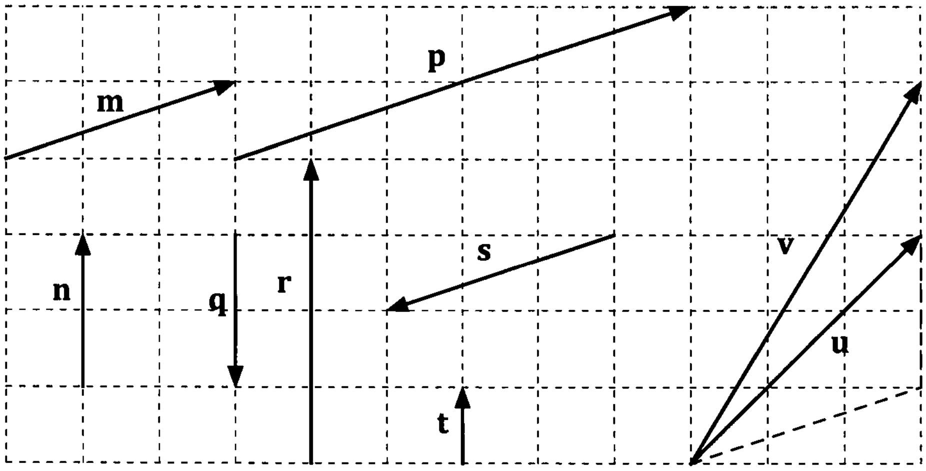
2. For each of the following diagrams select the appropriate vector equation from:
   a + **b** = **c**, **a** + **c** = **b**, **b** + **c** = **a**.



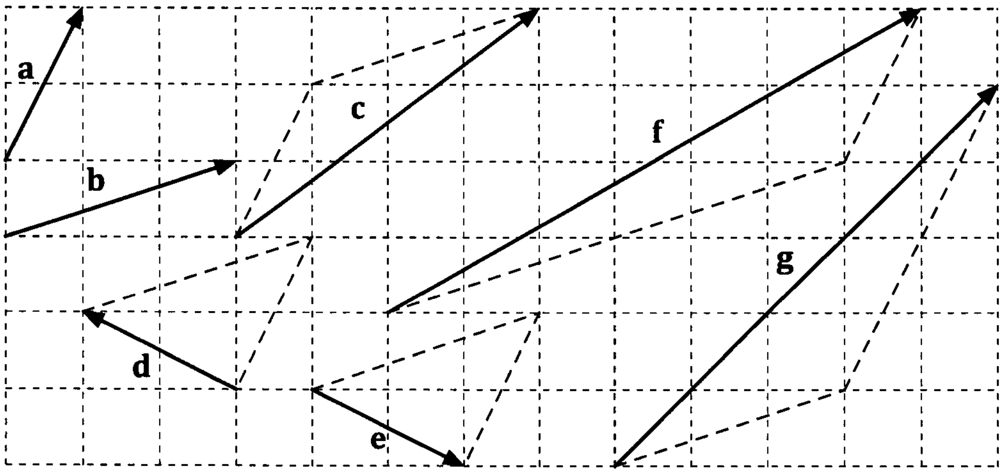
3. With **a** as defined in the diagram below express each of the vectors **b**, **c**, **d**, **e**, **f**, **g** and **h** in terms of **a**.



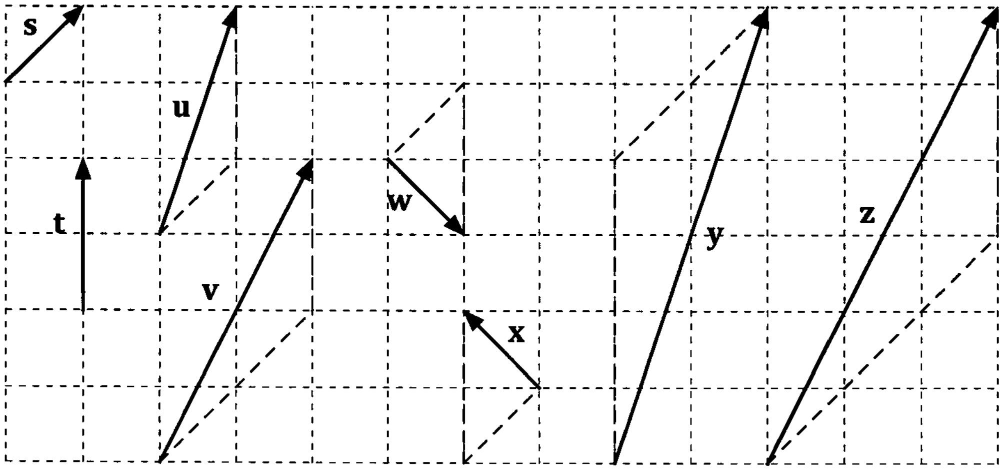
4. With **m** and **n** as defined in the diagram below express each of the vectors **p**, **q**, **r**, **s**, **t**, **u** and **v** in terms of **m** and/or **n**.



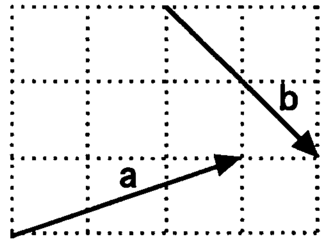
5. With **a** and **b** as defined in the diagram below express each of the vectors **c**, **d**, **e**, **f** and **g** in terms of **a** and **b**.



6. With **s** and **t** as defined in the diagram below express each of the vectors **u**, **v**, **w**, **x**, **y** and **z** in terms of **s** and **t**.



7. With **a** and **b** as defined in the diagram on the right draw on grid paper each of the following:



(a) 2**a**, (b) 3**b**,
(c) – **a**, (d) –**b**,
(e) **a** + **b**, (f) **a** – **b**,
(g) **b** – **a**, (h) **a** + 2**b**.

8. If **a** is a vector of magnitude 5 units in direction 070° and **b** is a vector of magnitude 4 units in direction 330° find the magnitude (correct to one decimal place) and direction (to the nearest degree) of the following.
(a) **a** + **b**, (b) **a** – **b**.

9. If **e** is a vector of magnitude 40 units in direction 130° and **f** is a vector of magnitude 30 units in direction 260° find the magnitude (correct to the nearest unit) and direction (to the nearest degree) of the following.
(a) 2**e** + **f**, (b) **e** – 2**f**.

10. The velocity of a body changes from **u**, 5·4 m/s due North, to **v**, 7·8 m/s due West, in 5 seconds. The acceleration of the body is given by $\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{\text{time taken}}$. Find the magnitude and direction of **a**.

11. The velocity of a body changes from an initial velocity **u**, 10·4 m/s in direction 020°, to a final velocity **v**, 12·1 m/s due West, in 4 seconds.
The body's acceleration is given by $\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{\text{time taken}}$.
Find the magnitude and direction of **a**.

12. Find the values of λ and μ in each of the following cases given that **a** and **b** are non-parallel vectors.
(a) $\lambda\mathbf{a} = \mu\mathbf{b}$, (b) $3\lambda\mathbf{a} = 5\mu\mathbf{b}$,
(c) $(\lambda - 3)\mathbf{a} = (\mu + 4)\mathbf{b}$, (d) $\lambda\mathbf{a} - 2\mathbf{a} = 5\mathbf{b} - \mu\mathbf{b}$,
(e) $\lambda\mathbf{a} - 2\mathbf{b} = \mu\mathbf{b} + 5\mathbf{a}$, (f) $(\lambda + \mu - 4)\mathbf{a} = (\mu - 3\lambda)\mathbf{b}$,
(g) $2\mathbf{a} + 3\mathbf{b} + \mu\mathbf{b} = 2\mathbf{b} + \lambda\mathbf{a}$, (h) $\lambda\mathbf{a} + \mu\mathbf{b} + 2\lambda\mathbf{b} = 5\mathbf{a} + 4\mathbf{b} + \mu\mathbf{a}$,
(i) $\lambda\mathbf{a} - \mathbf{b} + \mu\mathbf{b} = 4\mathbf{a} + \mu\mathbf{a} - 4\lambda\mathbf{b}$, (j) $2\lambda\mathbf{a} + 3\mu\mathbf{a} - \mu\mathbf{b} + 2\mathbf{b} = \lambda\mathbf{b} + 2\mathbf{a}$.

13. OABC is a rectangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. P and Q are the mid-points of AB and BC respectively. Express each of the following in terms of **a** and/or **c**.
(a) $\overrightarrow{CB}$, (b) $\overrightarrow{BC}$, (c) $\overrightarrow{AB}$, (d) $\overrightarrow{BA}$,
(e) $\overrightarrow{AP}$, (f) $\overrightarrow{OQ}$, (g) $\overrightarrow{OP}$, (h) $\overrightarrow{PQ}$.

14. OAB is a triangle with C a point on AB such that AC is three-quarters of AB.

    If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ express each of the following in terms of **a** and/or **b**.

    (a) $\overrightarrow{AB}$, (b) $\overrightarrow{AC}$, (c) $\overrightarrow{CB}$, (d) $\overrightarrow{OC}$.

15. ABCD is a parallelogram with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.
    E is a point on BC such that BE : EC = 1: 2.
    F is a point on CD such that CF : CD = 1: 2.
    Express each of the following in terms of **a** and/or **b**.

    (a) $\overrightarrow{AC}$, (b) $\overrightarrow{BE}$, (c) $\overrightarrow{DF}$, (d) $\overrightarrow{AE}$,

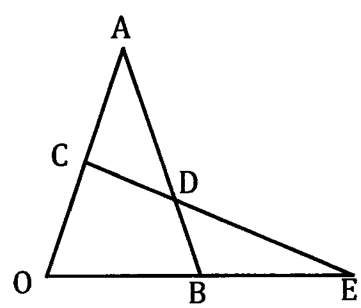
    (e) $\overrightarrow{AF}$, (f) $\overrightarrow{BF}$, (g) $\overrightarrow{DE}$, (h) $\overrightarrow{EF}$.

16. OABC is a trapezium with OC parallel to, and twice as long as, AB. D is the mid-point of BC.

    If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$ express each of the following in terms of **a** and /or **b**.

    (a) $\overrightarrow{OB}$, (b) $\overrightarrow{OC}$, (c) $\overrightarrow{BC}$, (d) $\overrightarrow{BD}$, (e) $\overrightarrow{OD}$.

17. OAB is a triangle with C the mid-point of OA and D a point on AB such that AD is two thirds of AB. CD continued meets OB continued at E.

    If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ express each of the following in terms of **a** and/or **b**.

    (a) $\overrightarrow{OC}$, (b) $\overrightarrow{AB}$, (c) $\overrightarrow{AD}$, (d) $\overrightarrow{CD}$,

    (e) If $\overrightarrow{CE} = h\overrightarrow{CD}$ and $\overrightarrow{OE} = k\overrightarrow{OB}$ use the fact that $\overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OE}$ to determine h and k.

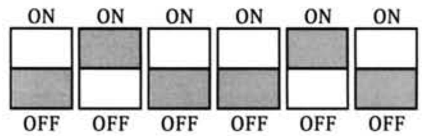
18. In the trapezium OABC, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{AB} = 2\mathbf{c}$. D is a point on CB such that $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{CB}$. OD continued meets AB continued at E.

    If $\overrightarrow{OE} = h\overrightarrow{OD}$ and $\overrightarrow{AE} = k\overrightarrow{AB}$ determine h and k.

Miscellaneous Exercise Three.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Each switch in a system of six switches can be either on or off.
   How many different settings are there for the system?



2. Jack walks 2·4 km on a bearing 060° followed by 4·4 km on a bearing 190°. On what bearing and for what distance should he now walk to return directly to his starting point?

3. Members of a wine club are invited to select a combination of eight different bottles of wine from a list of one dozen wines. How many different combinations are possible?

4. What is the least number of people needed to be in a room before we could guarantee that at least three of them would have the same birth month?

5. Let us suppose that for the purposes of this question I teach Specialist Mathematics at, and only at, XYZ high school, a year 7 to 12 high school.
   Under the above assumption the following would be a true statement:
   If you are in my Specialist Mathematics class
   then you attend XYZ high school.
   Write the converse of this statement and the contrapositive of this statement and in each case state whether true or false.

6. Eight members of a rugby team are to be arranged in a line for a photograph. If the eight can themselves be chosen from the fifteen in the team how many different possible arrangements are there?

7. If $\mathbf{c}$ is a vector of magnitude 10 units in direction 160° and $\mathbf{d}$ is a vector of magnitude 12 units due East find the magnitude (correct to one decimal place) and direction (to the nearest degree) of the following.
   (a) $\mathbf{c} + \mathbf{d}$, (b) $\mathbf{c} - \mathbf{d}$, (c) $\mathbf{c} + 2\mathbf{d}$.

8. Given that $\mathbf{a}$ and $\mathbf{b}$ are not parallel determine h and k in each of the following vector statements.
   (a) $h\mathbf{a} = k\mathbf{b}$ (b) $h\mathbf{a} + \mathbf{b} = k\mathbf{b}$
   (c) $(h-3)\mathbf{a} = (k+1)\mathbf{b}$ (d) $h\mathbf{a} + 2\mathbf{a} = k\mathbf{b} - 3\mathbf{a}$
   (e) $3h\mathbf{a} + k\mathbf{a} + h\mathbf{b} - 2k\mathbf{b} = \mathbf{a} + 5\mathbf{b}$ (f) $h(\mathbf{a} + \mathbf{b}) + k(\mathbf{a} - \mathbf{b}) = 3\mathbf{a} + 5\mathbf{b}$

9. A group of 5 people is to be formed from a group comprising 10 women and 6 men. How many different groups of 5 people are there if the group must consist of
   (a) 3 women and 2 men, (b) all women,
   (c) more women than men, (d) more men than women?